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On lateral–torsional vibrations of elastic composite beams with interlayer slip

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A B S T R A C T

The lateral–torsional vibrations of composite beams are investigated in this paper, based on a variational approach. Composite beams studied in this paper are classified as composite beams with interlayer slip (composite beams with partial interaction or layered wood beams) or three-layer sandwich beams. It will be shown in this paper that both structural systems are governed by the same nature of differential equations. The theoretical framework of lateral–torsional vibrations of composite beams is given, and some engineering results are presented for the pinned–pinned strip composite beam. A simple closed-form solution is achieved for the lateral–torsional natural frequencies. The results are analogous to the ones obtained for the in-plane vibrations of composite beams (sandwich beam or composite beams with partial interaction) where the natural frequencies increase with the stiffness of the connection. Extension of these results to some more complex loading cases is envisaged, and a numerical procedure will be probably needed in the general case.

1. Introduction

Composite structures of different materials have important applications in civil or mechanical engineering. The out-of-plane vibrations of partially composite beams are investigated in this paper, based on a variational approach. The theoretical framework of lateral–torsional vibrations of composite beams is given, and some engineering results are presented. Composite beams studied in this paper are classified as composite beams with interlayer slip (composite beams with partial interaction or layered wood beams) or three-layer sandwich beams. It is shown in this paper that both structural systems are governed by the same nature of differential equations.

Layered composite structural elements are typically encountered in wood design, where wood beams are made from layers assembled by means of nailing, bolting or gluing. Composite structures of different materials such as timber-concrete elements or steel-concrete elements are widely used in building engineering. These composite structures are built up by structural subelements connected by shear connectors to form an interacting unit. In the case of a flexible connection, the analysis procedure requires consideration of the interlayer slip between the subelements, leading to the partial interaction concept. The fundamental equations of the theory of layer wood beams or composite structures with partial interaction were developed by Stüssi [1], Granholm [2], Newmark et al [3] and Pleshkov [4] (see also Ref. [5] or the recent paper on partially composite beams of Girhammar and Pan [6]). The in-plane vibrations of elastic composite beams with interlayer slip are investigated by Henghold [7] or Girhammar and Pan [8] who used Euler–Bernoulli beam models for

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each beam (see also Refs. [9,10]). Berczynski and Wroblewski [11] generalize the results of Girhammar and Pan [8] obtained for Euler–Bernoulli beam models, by using Timoshenko's beam theory (see also Ref. [12] for the treatment of general boundary conditions including the in-plane stability problem). Dilena and Morassi [13] recently considered the case of a continuously damaged connection for the in-plane vibrations of partially composite beams.

Three-layer sandwich beams, on the other hand, may be composed of two identical thin face layers and a thick core layer [14,15]. The obvious advantage of this construction is the large moment of inertia of the section obtained by spacing far apart the main carrying elements, namely, the faces. The weight of the structure is small because of the low density of the core. The key point in using sandwich structures is the possibility of largely reducing weights while keeping the same equivalent stiffnesses. These sandwich beams have been widely studied in the 1960s especially for their damping dynamic characteristics [16–19]. These authors used a three-layer sandwich beam theory based on Euler–Bernoulli model for each beam. Mead [20] or Chonan [21] extended these results incorporating the effect of shear deformation of each beam. All these studies are devoted to the in-plane vibrations of sandwich beams. Another application of sandwich structural models can be found in building engineering, where high-rise buildings can be modelled by equivalent sandwich beams (see for instance Ref. [22]). As shown by Heuer [23] (see also the pioneer paper of McCutcheon [24]), there is a correspondence between the three-layer sandwich beam and the composite beam with partial interaction. Such a correspondence for the in-plane vibrations problem can be easily extended for the out-of-plane vibrations problem (see also Challamel and Girhammar [25] for the correspondence of the lateral–torsional buckling problem).

Even if the lateral–torsional vibrations of composite beams have been extensively studied (see for instance Ref. [26]), very few works have been devoted to the out-of-plane vibrations of beams with interlayer slip (or sandwich beams). The recent work of Numayr and Qablan [27] is probably an exception for the free vibrations of sandwich beams. We will present a slightly different theory, specifically for the torsion–bending coupling in the soft core. There is undoubtedly a need to elaborate an engineering theory for lateral–torsional vibrations of composite members with partial interaction in simple structural cases.

2. Energy equations

For the layered or partial composite beam, the cross-section is subdivided into two parts "1" and "2" with regular boundary along a rectilinear axis orthogonal to the cross-section (see Fig. 1). The kinematic assumptions are similar to the ones chosen by Dall'Asta [28]. A discontinuity of displacement field can occur at the connection plane (see Fig. 1). In the theory presented, it is assumed that the connection, denoted as shear connection, does not permit a displacement jump in the direction orthogonal to the connection plane. In other words, the twisting rotation φ must be the same for the two components. This fundamental assumption related to the kinematics of the present model means that only horizontal slip along the discontinuity line is allowed (see Fig. 1). A possible generalization of this kinematics could be the coupling between a vertical and a horizontal displacement jump (and why not a rotation), following the pioneer study of Adekola [29] restricted to the in-plane behaviour of partially composite beam [29].

The deflection at the onset of buckling is specified by (i) the cross-section rotation angle $\varphi(x)$ in the yz plane, (ii) the displacement $w_1(x)$ of the beam axis of the domain "1" in the z direction (h_1 represents the depth of layer 1) and (iii) the displacement $w_2(x)$ of the beam axis of the domain "2" in the z direction (h_2 represents the depth of layer 2). Each subdomain is assumed to be composed of a thin strip beam of width b_i (see Fig. 2a). For sandwich beams, it is often assumed that the width of each beam is identical, i.e. b_i is equal to b. The total potential energy π is written as the sum of three terms:

$$\pi[\varphi, w_1, w_2] = \pi_1[\varphi, w_1, w_2] + \pi_2[\varphi, w_1, w_2] + \pi_3[\varphi, w_1, w_2] \tag{1}$$

where π_1 is the potential energy associated to the subdomain "1", π_2 is the potential energy associated to the subdomain "2", and π_3 is the potential energy associated to the connection. The independence of the in-plane and out-of-plane motion is implicitly postulated in such a formulation. The first term is written as:

$$\pi_1[\varphi, w_1, w_2] = \int_0^L \frac{1}{2} GJ_1[\varphi'(x)]^2 + \frac{1}{2} EI_1[w_1''(x)]^2 \, dx \tag{2}$$

where GJ_1 is the beam torsional stiffness of the subdomain "1" and EI_1 is the beam stiffness in the xz plane of the subdomain "1" (which can be also denoted by EI_{y1}). GJ_1 (respectively, EI_1) can be considered as the condensed notation of G_1J_1 (respectively, E_1I_1), or $(GJ)_1$ (respectively, $(EI)_1$) in case of different elastic moduli for both parts of the cross-section. L

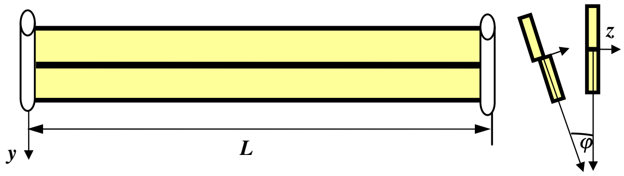

Fig. 1. Geometry of the hinged–hinged strip beam.

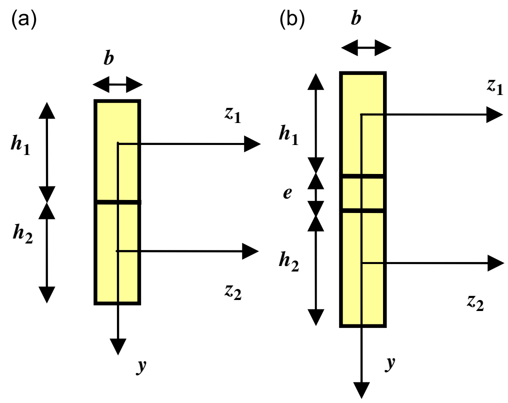

Fig. 2. Characteristics of the cross-section. (a) Layered wood beams or partial composite beam. (b) Three-layer sandwich beam.

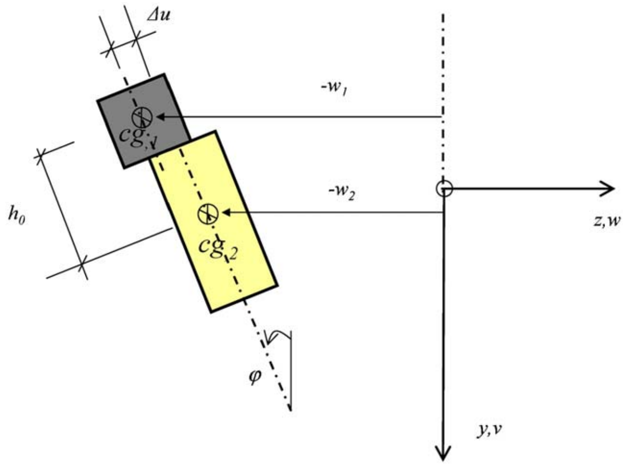

Fig. 3. Kinematics of the partially composite section.

is the length of the beam. The second term of the potential energy is similar to the first one:

$$\pi_2[\varphi, w_1, w_2] = \int_0^L \frac{1}{2} GJ_2[\varphi'(x)]^2 + \frac{1}{2} EI_2[w_2''(x)]^2 \, dx \tag{3}$$

where GJ_2 is the beam torsional stiffness of the subdomain "2" and EI_2 is the beam stiffness in the xz plane of the subdomain "2" (which can be also denoted by EI_{y2}). GJ_2 (respectively, EI_2) can be also considered as the condensed notation of G_2J_2 (respectively, E_2I_2), or $(GJ)_2$ (respectively, $(EI)_2$) in case of different elastic moduli for both parts of the cross-section. Finally, the last term of the total potential energy is written for the composite beam with partial interaction as

$$\pi_3[\varphi, w_1, w_2] = \int_0^L \frac{1}{2} K[\Delta u(\varphi, w_1, w_2)]^2 \, dx \quad \text{with } \Delta u = w_2 - w_1 - h_0\varphi \text{ and } h_0 = \frac{h_1 + h_2}{2} \tag{4}$$

The kinematics of the partially composite section with horizontal discontinuity line is shown in Fig. 3. It can be seen from Eq. (4) that the interface slip Δu depends on both the transversal deflections w_i, but also on the torsional angle φ. This dependence leads to the full coupling between lateral and torsional vibrations. The interface slip Δu has been calculated from geometrical considerations based on the assumption of small torsional angle. The connector load-slip behaviour is linear elastic with a constant slip modulus K. Full composite action (infinite slip stiffness, $K \to \infty$) and non-composite action (zero slip stiffness, $K \to 0$) represent upper and lower bounds for the partial composite action. In the case of horizontal discontinuity (Fig. 3), the slip forces are due to torsion and to the difference in curvature with respect to the weak axis.

For the three layer sandwich beam (Fig. 2b), the total potential energy π is also the sum of three terms (Eq. (1) is still valid), but the total potential energy of the soft core is equal to

$$\pi_3[\varphi, w_1, w_2] = \int_0^L \frac{1}{2} beG^* \gamma^2 \, dx \quad \text{with } \gamma = \frac{w_2 - w_1 - h_0\varphi}{e} \text{ and } h_0 = e + \frac{h_1 + h_2}{2} \tag{5}$$

where G^* is the shear modulus of the soft core and γ is the shear strain in the soft interlayer. The comparison of Eqs. (4) and (5) shows that there is equivalence between both structural systems with the following identities:

$$\frac{bG^*}{e} = K \quad \text{and} \quad h_0 = e + \frac{h_1 + h_2}{2} \tag{6}$$

The layered or partial composite beam is obtained as a particular case with a vanishing depth of the interlayer e. Therefore, the partial composite beam and the three-layer sandwich beam are strictly equivalent structural problems. The same conclusion holds for the in-plane bending problem (see for instance Refs. [23,24]). With the substitution according to Eq. (6), all results obtained in this study are also applicable to sandwich-type of beams.

It is chosen to make the presentation with the partial composite notation of Eq. (4). The kinetic energy is equal to

$$T[\varphi, w_1, w_2] = \int_0^L \frac{1}{2} m_1 \dot{w}_1^2 + \frac{1}{2} m_2 \dot{w}_2^2 + \frac{1}{2} m_0 r_0^2 \dot{\varphi}^2 \, dx \quad \text{with } m_0 r_0^2 = m_1 r_1^2 + m_2 r_2^2 \tag{7}$$

where m_1 (respectively, m_2) is the mass per unit length of the subdomain "1" (respectively, "2"), and r_1 (respectively, r_2) is the cross-sectional mass radius of gyration of the subdomain "1" (respectively, "2"). The dynamics equations are obtained via the Hamilton principle, leading to the partial-differential equations:

$$\begin{cases} EI_1 w_1^{(4)} - K(w_2 - w_1 - h_0\varphi) + m_1 \ddot{w}_1 = 0 \\ EI_2 w_2^{(4)} + K(w_2 - w_1 - h_0\varphi) + m_2 \ddot{w}_2 = 0 \\ -GJ_1 \varphi'' - GJ_2 \varphi'' - Kh_0(w_2 - w_1 - h_0\varphi) + m_0 r_0^2 \ddot{\varphi} = 0 \end{cases} \tag{8}$$

The free vibrations problem is investigated in this paper, based on the following system:

$$\begin{cases} EI_1 w_1^{(4)} - K(w_2 - w_1 - h_0\varphi) + m_1 \ddot{w}_1 = 0 \\ EI_2 w_2^{(4)} + K(w_2 - w_1 - h_0\varphi) + m_2 \ddot{w}_2 = 0 \\ GJ_0 \varphi'' + Kh_0(w_2 - w_1 - h_0\varphi) - m_0 r_0^2 \ddot{\varphi} = 0 \end{cases} \quad \text{with } GJ_1 + GJ_2 = GJ_0 \tag{9}$$

Note the fundamental difference with the model of Numayr and Qablan [27] who postulated a possible twist angle difference between each component ($\varphi_1 \neq \varphi_2$). However, even in the case where the twist angles of each component are identical ($\varphi_1 = \varphi_2 = \varphi$), as postulated in the present paper, the model investigated in Ref. [27] and the present model coincide, only if the parameter h_0 is vanishing ($h_0 = 0$). Therefore, the bending–torsional coupling of the present model is quite different from the one studied in Ref. [27] in the soft core.

In the general case, for arbitrary boundary conditions, a numerical method is needed to solve the vibrations problem. One of the most popular methods for the approximate integration of differential equations is the Bubnov–Galerkin method (see for instance Ref. [30]). The finite difference method is employed by Numayr and Qablan [27] for the out-of-plane vibrations of sandwich beams. Such numerical methods are not useful in the present paper, as closed-form solutions will be available for some specific boundary conditions.

3. Free vibrations

The lateral–torsional vibrations of the pinned–pinned strip beam are studied in this paper. The boundary conditions derived from application of Hamilton's principle can be expressed in this case as

$$\varphi(0) = \varphi(L) = 0, \quad w_1''(0) = w_1''(L) = w_2''(0) = w_2''(L) = 0, \quad w_1(0) = w_1(L) = w_2(0) = w_2(L) = 0 \tag{10}$$

The solution is sought in the form

$$\begin{cases} w_1(x,t) = w_1^0 \sin\dfrac{n\pi x}{L} \sin \Omega t \\ w_2(x,t) = w_2^0 \sin\dfrac{n\pi x}{L} \sin \Omega t \\ \varphi(x,t) = \varphi^0 \sin\dfrac{n\pi x}{L} \sin \Omega t \end{cases} \tag{11}$$

Introducing this solution in the partial-differential equation (9) leads to the characteristic equation:

$$\begin{vmatrix} EI_1\left(\frac{n\pi}{L}\right)^4 + K - m_1\Omega^2 & -K & Kh_0 \\ -K & EI_2\left(\frac{n\pi}{L}\right)^4 + K - m_2\Omega^2 & -Kh_0 \\ Kh_0 & -Kh_0 & GJ_0\left(\frac{n\pi}{L}\right)^2 + Kh_0^2 - m_0 r_0^2\Omega^2 \end{vmatrix} = 0 \tag{12}$$

In the case of non-composite action ($K = 0$), the uncoupled vibration frequencies ($\Omega_1, \Omega_2, \Omega_3$) can be obtained:

$$\Omega_1^2 = \frac{EI_1}{m_1}\left(\frac{n\pi}{L}\right)^4, \quad \Omega_2^2 = \frac{EI_2}{m_2}\left(\frac{n\pi}{L}\right)^4 \quad \text{and} \quad \Omega_3^2 = \frac{GJ_0}{m_0 r_0^2}\left(\frac{n\pi}{L}\right)^2 \tag{13}$$

The first frequencies Ω_1 and Ω_2 are associated to a flexural vibrations mode of each component, whereas the last frequency Ω_3 corresponds to a pure torsional mode. In the general case, however, in the presence of composite action ($K \neq 0$), the flexural–torsional vibration modes are strongly coupled. This phenomenon is similar to the one observed for beams with unsymmetrical cross-sections, where the beam undergoes combined flexural–torsional vibrations [26,30,31]. The characteristic equation can be simplified as

$$\left(EI_1\left(\frac{n\pi}{L}\right)^4 - m_1\Omega^2\right)\left(EI_2\left(\frac{n\pi}{L}\right)^4 - m_2\Omega^2\right)\left(GJ_0\left(\frac{n\pi}{L}\right)^2 + Kh_0^2 - m_0 r_0^2\Omega^2\right) + K\left(GJ_0\left(\frac{n\pi}{L}\right)^2 - m_0 r_0^2\Omega^2\right)\left(EI_0\left(\frac{n\pi}{L}\right)^4 - m_0\Omega^2\right) = 0 \quad \text{with } EI_1 + EI_2 = EI_0 \text{ and } m_1 + m_2 = m_0 \tag{14}$$

Note that the introduction of the warping terms for symmetrical partially composite thin-walled beams can be easily included (see Appendix A).

4. Resolution—dimensionless formulation

The dimensionless parameters are introduced as

$$\omega^2 = \frac{\Omega^2 m_0 L^4}{EI_0}, \quad \lambda = \frac{EI_0}{GJ_0}, \quad \bar{r} = \frac{r_0}{L}, \quad \mu_1 = \frac{m_1}{m_0}, \quad \mu_2 = \frac{m_2}{m_0}, \quad k_1 = \frac{EI_1}{EI_0}, \quad k_2 = \frac{EI_2}{EI_0}$$

$$\kappa = KL^4\left(\frac{1}{EI_1} + \frac{1}{EI_2}\right) = KL^4\frac{EI_0}{EI_1 EI_2} \quad \text{and} \quad \alpha = \left(\frac{h_0}{L}\right)^2\frac{EI_1 EI_2}{EI_0 GJ_0} \tag{15}$$

Note that some parameters are linked by the rule:

$$k_1 + k_2 = 1 \quad \text{and} \quad \mu_1 + \mu_2 = 1 \tag{16}$$

The nonlinear frequency equation is then written with the dimensionless variables:

$$(k_1(n\pi)^4 - \mu_1\omega^2)(k_2(n\pi)^4 - \mu_2\omega^2)((n\pi)^2 + \kappa\alpha - \bar{r}^2\lambda\omega^2) + \kappa k_1 k_2((n\pi)^2 - \bar{r}^2\lambda\omega^2)((n\pi)^4 - \omega^2) = 0 \tag{17}$$

In the case of non-composite action ($\kappa = 0$), the uncoupled vibrations frequencies are found again:

$$\omega_1^2 = \frac{k_1}{\mu_1}(n\pi)^4, \quad \omega_2^2 = \frac{k_2}{\mu_2}(n\pi)^4 \quad \text{and} \quad \omega_3^2 = \frac{1}{\lambda\bar{r}^2}(n\pi)^2 \tag{18}$$

Eq. (17) can also be written as a third-order polynomial equation which can be solved using Cardano's method [32]:

$$ax^3 + bx^2 + cx + d = 0 \quad \text{with } x = \omega^2 \tag{19}$$

The constants are identified from Eq. (17) as

$$\begin{cases} a = -\mu_1\mu_2\bar{r}^2\lambda \\ b = \bar{r}^2\lambda(k_2\mu_1 + k_1\mu_2)(n\pi)^4 + \mu_1\mu_2((n\pi)^2 + \kappa\alpha) + \kappa k_1 k_2\bar{r}^2\lambda \\ c = -[k_1k_2(n\pi)^8\bar{r}^2\lambda + (k_2\mu_1 + k_1\mu_2)(n\pi)^4((n\pi)^2 + \kappa\alpha) + \kappa k_1 k_2(n\pi)^2(1 + \bar{r}^2\lambda(n\pi)^2)] \\ d = k_1k_2(n\pi)^6[(n\pi)^2((n\pi)^2 + \kappa\alpha) + \kappa] \end{cases} \tag{20}$$

Eq. (19) can be written in the canonical form

$$y^3 + py + q = 0 \quad \text{with } y = x + \frac{b}{3a}, \ p = \frac{3ac - b^2}{3a^2} \text{ and } q = \frac{27a^2d + 2b^3 - 9abc}{27a^3} \tag{21}$$

It can be numerically checked that generally $4p^3+27q^2<0$, and then, the three solutions are given by

$$y_1 = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)+2\pi}{3}\right],\quad y_2 = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)+4\pi}{3}\right]$$

and

$$y_3 = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)}{3}\right] \tag{22}$$

Therefore, the frequency solutions are obtained as

$$\omega_1^2 = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)+2\pi}{3}\right]-\frac{b}{3a}$$

$$\omega_2^2 = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)+4\pi}{3}\right]-\frac{b}{3a}$$

and

$$\omega_3^2 = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{\arccos\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right)}{3}\right]-\frac{b}{3a} \tag{23}$$

The evolution of the natural frequencies of each fundamental mode ($n=1$) is shown in Fig. 4 and 5 for two unsymmetrical cases. The sensitivity of the natural frequencies to the dimensionless parameters is discussed thereafter. For strip beams, the stiffness ratio λ_1 and λ_2 are calculated as

$$\lambda_1 = \frac{EI_1}{GJ_1} = \frac{1+\upsilon_1}{2} \quad \text{and} \quad \lambda_2 = \frac{EI_2}{GJ_2} = \frac{1+\upsilon_2}{2} \tag{24}$$

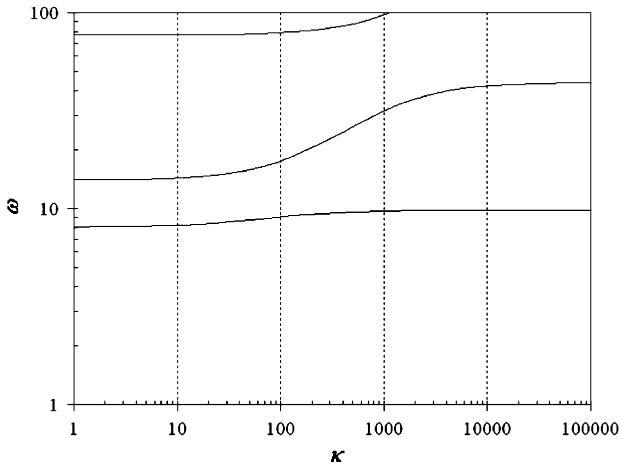

Fig. 4. Evolution of the frequencies of each fundamental mode versus the connection parameter κ—$\omega_1<\omega_2<\omega_3$; unsymmetrical case: $k_1=k_2=0.5$, $\mu_1=0.75$, $\mu_2=0.25$, $\alpha=5\times10^{-3}$, $\lambda\bar{r}^2=\alpha/3$.

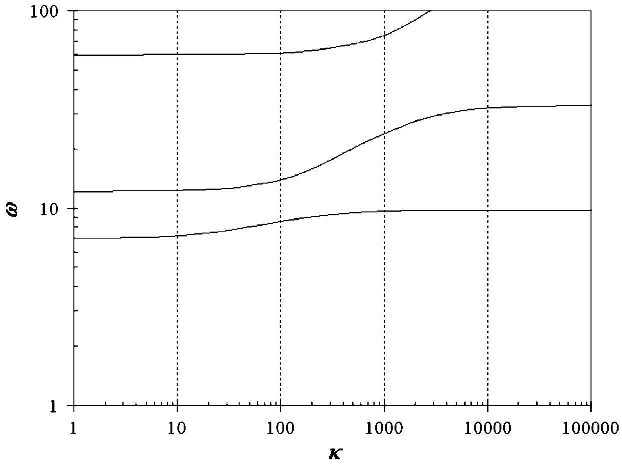

Fig. 5. Evolution of the frequencies of each fundamental mode versus the connection parameter κ—$\omega_1 < \omega_2 < \omega_3$; unsymmetrical case: $\mu_1 = \mu_2 = 0.5$, $k_1 = 0.75$, $k_2 = 0.25$, $\alpha = 5 \times 10^{-3}$, $\lambda \bar{r}^2 = 5\alpha/9$.

where υ_1 denoted the Poisson's ratio of subelement "1", whereas υ_2 denoted the Poisson's ratio of subelement "2". For instance, the global stiffness ratio can simplified for homogeneous subelements as

$$\lambda_1 = \lambda_2 = \frac{1+\upsilon}{2} \Rightarrow \lambda = \frac{EI_0}{GJ_0} = \frac{\lambda_1 GJ_1 + \lambda_2 GJ_2}{GJ_1 + GJ_2} = \frac{1+\upsilon}{2} \tag{25}$$

where υ is Poisson's ratio of each subelement. Parameter α can be also expressed in terms of other dimensionless parameters:

$$\alpha = \frac{k_1 k_2 \lambda}{4}\left(\frac{h}{L}\right)^2 \quad \text{with } h = h_1 + h_2 \tag{26}$$

Therefore the order of magnitude of the parameter α is typically comprises in the order of 10^{-4} or 10^{-3}. The dimensionless equivalent radius of gyration $\bar{r}$ is calculated for the composite strip beam as

$$\bar{r}^2 = \frac{1}{12}\left(\mu_1\left(\frac{h_1}{L}\right)^2 + \mu_2\left(\frac{h_2}{L}\right)^2\right) \tag{27}$$

Furthermore, for constant width ($b_1 = b_2$) and for homogeneous subelements ($E_1 = E_2$), it is easily shown that the stiffness ratio k_1 and k_2 are dependent on the height ratio via:

$$k_1 = \frac{h_1}{h} \quad \text{and} \quad k_2 = \frac{h_2}{h} \tag{28}$$

Hence, there is a strong relationship between parameters α and $\lambda\bar{r}^2$, obtained from Eqs. (26) and (27), leading to

$$\frac{\lambda\bar{r}^2}{\alpha} = \frac{1}{3}\frac{\mu_1 k_1^2 + \mu_2 k_2^2}{k_1 k_2} \tag{29}$$

Some particular cases can be deduced from Eq. (29). The case of a layered wood beam with two identical subelements leads to

$$\mu_1 = \mu_2 = k_1 = k_2 = \frac{1}{2} \Rightarrow \frac{\lambda\bar{r}^2}{\alpha} = \frac{1}{3} \tag{30}$$

This is also the value obtained in the more general case of two subelements with the same stiffness:

$$\mu_1 \neq \mu_2, \quad k_1 = k_2 = \frac{1}{2} \Rightarrow \frac{\lambda\bar{r}^2}{\alpha} = \frac{1}{3} \tag{31}$$

Finally, we give another example of unsymmetrical section, based on

$$\mu_1 = \mu_2 = \frac{1}{2}, \quad k_1 = \frac{3}{4}, \quad k_2 = \frac{1}{4} \Rightarrow \frac{\lambda\bar{r}^2}{\alpha} = \frac{5}{3}\frac{1}{3} \tag{32}$$

Figs. 4 and 5 show the sensitivity of the natural frequencies with respect to the connection parameter, for two unsymmetrical beams. Fig. 4 corresponds to a beam with unsymmetrical inertia terms, whereas Fig. 5 is associated to unsymmetrical stiffness terms. The shape of the frequency curve with respect to the dimensionless connection parameter

κ is close to the one observed for the in-plane partially composite vibration problem (see Ref. [8] for the partially composite Euler–Bernoulli beams, or Ref. [12] for the partially composite Timoshenko beams). The frequencies grow with the stiffness connection and tend towards a finite frequency value for the two lowest mode of the full-composite beam ($\kappa \to \infty$). However, the third mode (predominant torsional mode) gives infinite frequencies when the dimensionless connection parameter κ tends towards an infinite value (see Figs. 4 and 5).

5. Case with two identical subelements

The shape of the frequency curve will be investigated with further details in the particular symmetrical case, with two identical subelements. The order of magnitude of parameter α can be evaluated for a layered wood beam with two identical subelements ($EI_1 = EI_2 = EI_0/2$):

$$EI_1 = EI_2 = \frac{EI_0}{2} \;\Rightarrow\; \alpha = \left(\frac{h_0}{L}\right)^2 \frac{EI_0}{4GJ_0} = \left(\frac{h}{L}\right)^2 \frac{1+\upsilon}{32} \quad \text{with } h = h_1 + h_2 \tag{33}$$

where υ is Poisson's ratio. For the beam with two identical subelements, the following equalities can be assumed:

$$\mu_1 = \mu_2 = k_1 = k_2 = \frac{1}{2} \;\Rightarrow\; \bar{r}^2 = \frac{1}{48}\left(\frac{h}{L}\right)^2 \;\Rightarrow\; \lambda\bar{r}^2 = \frac{\alpha}{3} \tag{34}$$

In the case of the beam with two identical subelements, the natural frequency equation Eq. (17) is factorized by

$$((n\pi)^4 - \omega^2)[((n\pi)^4 - \omega^2)((n\pi)^2 + \kappa\alpha - \bar{r}^2\lambda\omega^2) + \kappa((n\pi)^2 - \bar{r}^2\lambda\omega^2)] = 0 \tag{35}$$

The second-order polynomial term is developed as

$$\lambda\bar{r}^2\omega^4 - (\lambda\bar{r}^2(n\pi)^4 + (n\pi)^2 + \kappa(\alpha + \lambda\bar{r}^2))\omega^2 + (n\pi)^4((n\pi)^2 + \kappa\alpha) + \kappa(n\pi)^2 = 0 \tag{36}$$

whose solution is finally expressed as

$$\omega_1^2 = (n\pi)^4$$

$$\omega_2^2 = \frac{\lambda\bar{r}^2(n\pi)^4 + \kappa\alpha + (n\pi)^2 + \kappa\lambda\bar{r}^2 - \sqrt{[\lambda\bar{r}^2(n\pi)^4 + \kappa\alpha + (n\pi)^2 + \kappa\lambda\bar{r}^2]^2 - 4\lambda\bar{r}^2[(n\pi)^6 + \kappa\alpha(n\pi)^4 + \kappa(n\pi)^2]}}{2\lambda\bar{r}^2}$$

and

$$\omega_3^2 = \frac{\lambda\bar{r}^2(n\pi)^4 + \kappa\alpha + (n\pi)^2 + \kappa\lambda\bar{r}^2 + \sqrt{[\lambda\bar{r}^2(n\pi)^4 + \kappa\alpha + (n\pi)^2 + \kappa\lambda\bar{r}^2]^2 - 4\lambda\bar{r}^2[(n\pi)^6 + \kappa\alpha(n\pi)^4 + \kappa(n\pi)^2]}}{2\lambda\bar{r}^2} \tag{37}$$

In this last case, the asymptotic values associated to the full-composite section is simplified as

$$\kappa \to \infty \Rightarrow \omega_1^2 = (n\pi)^4, \quad \omega_2^2 \to \frac{(n\pi)^2}{\alpha + \lambda\bar{r}^2}(1 + \alpha(n\pi)^2) \approx \frac{(n\pi)^2}{\alpha + \lambda\bar{r}^2} \quad \text{and} \quad \omega_3^2 \to \kappa\frac{\alpha + \lambda\bar{r}^2}{\lambda\bar{r}^2} \tag{38}$$

In this symmetrical case, the two lowest modes coalesce for the non-composite beam ($\kappa = 0$), which is no more the case for the partially composite beam (see Figs. 6–8). The third mode gives infinite frequencies when the dimensionless connection

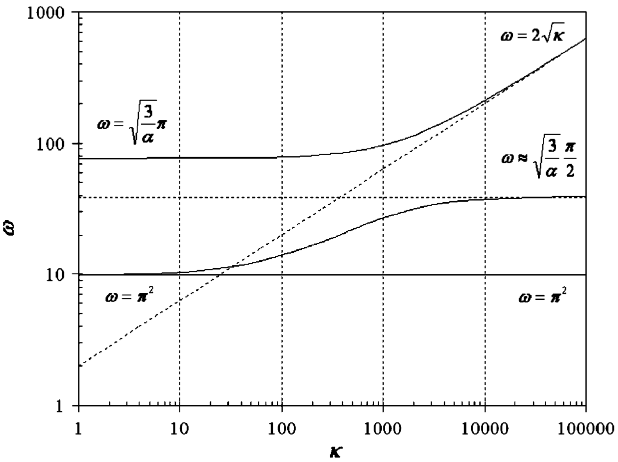

Fig. 6. Evolution of the frequencies versus the connection parameter κ—symmetrical case: $\mu_1 = \mu_2 = k_1 = k_2 = 0.5$, $\alpha = 5 \times 10^{-3}$, $\lambda\bar{r}^2 = \alpha/3$.

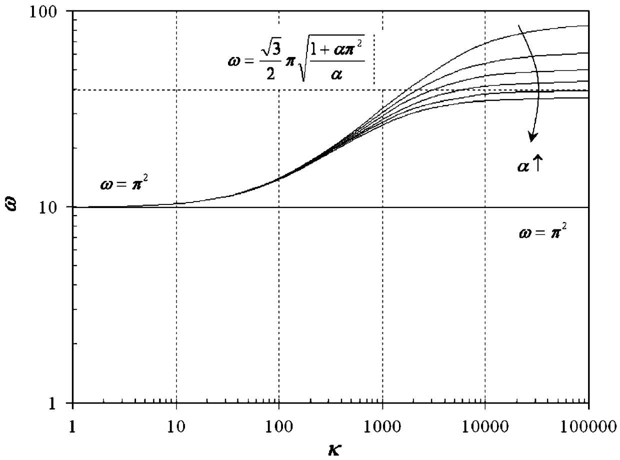

Fig. 7. Evolution of the frequencies of the two lowest mode versus the connection parameter κ—symmetrical case; $\alpha \in \{10^{-3}; 2\times10^{-3}; 3\times10^{-3}; 4\times10^{-3}; 5\times10^{-3}; 6\times10^{-3}\}$.

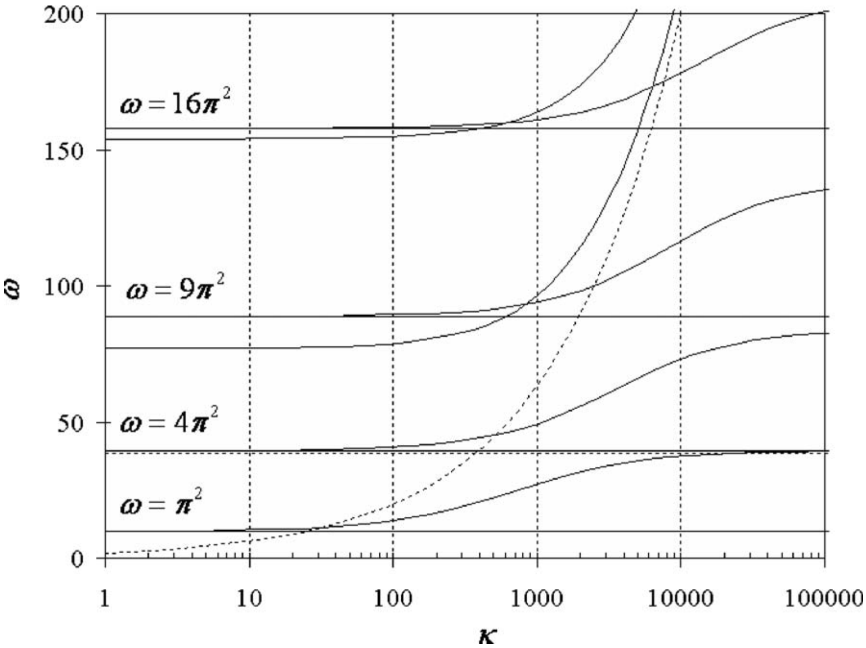

Fig. 8. Natural frequencies spectra versus the connection parameter κ—symmetrical case: $\alpha = 5\times10^{-3}$, $\lambda\bar{r}^2 = \alpha/3$.

parameter κ tends towards an infinite value. According to Eq. (34), this last predominantly torsional frequency is asymptotically given by

$$\lambda\bar{r}^2 = \frac{\alpha}{3}, \quad \kappa \to \infty \;\Rightarrow\; \omega_3^2 \to 4\kappa \tag{39}$$

The results presented in Figs. 6–8 show the dimensionless frequencies with respect to the connection parameter κ. The frequencies clearly increase with the stiffness of the connection, and the frequency spectra of the full-composite beam have been simply evaluated from an asymptotic method:

$$\lambda\bar{r}^2 = \frac{\alpha}{3}, \quad \kappa \to \infty \;\Rightarrow\; \omega_1^2 = (n\pi)^4, \quad \omega_2^2 \to \frac{3}{4}\frac{(n\pi)^2}{\alpha}(1+\alpha(n\pi)^2) \approx \frac{3}{4}\frac{(n\pi)^2}{\alpha} \tag{40}$$

The finite character of the natural frequencies in the asymptotic case ($\kappa \to \infty$) (see Eq. (40)) cannot be obtained when the parameter α is vanishing, as implicitly assumed for instance in Ref. [27]. It is worth mentioning that the shape of the frequencies curve obtained in these figures is very similar to the one obtained for the in-plane vibrations problem, except that the predominantly torsional mode has no finite threshold in case of the full composite beam ($\kappa \to \infty$). It has been shown that this torsional frequency is proportional to the square root of the dimensionless connection parameter κ for large values of this parameter.

6. Conclusions

The lateral–torsional vibrations of composite beams are investigated in this paper, based on a variational approach. Composite beams studied in this paper are classified as composite beams with interlayer slip (composite beams with partial interaction or layered wood beams) or three-layer sandwich beams. It has been shown in this paper that both structural systems are governed by the same nature of differential equations. The theoretical framework of lateral–torsional vibrations of composite beams is given, based on the independence of the in-plane and out-of-plane motion. Some engineering results are presented for the pinned–pinned strip composite beam. A simple closed-form solution is achieved for the lateral–torsional natural frequencies. The results are analogous to the ones obtained for the in-plane vibrations of composite beams (sandwich beam or composite beams with partial interaction) where the natural frequencies increase with the stiffness of the connection. It is worth mentioning that the shape of the frequencies curve obtained in these figures is very similar to the one obtained for the in-plane vibrations problem, except that the predominantly torsional mode has no finite threshold in case of the full composite beam ($\kappa \to \infty$). It has been shown that this torsional frequency is proportional to the square root of the dimensionless connection parameter κ for large values of this parameter. The other natural frequencies remain finite in case of the full-composite beams, as also observed for the in-plane vibrations problem. Extension of these results to some more complex loading cases is envisaged, and a numerical procedure will be probably needed in the general case.

Appendix A. Introduction of the warping effect

For symmetrical partially composite thin-walled beams, the warping terms may be easily added in the energy functional π_1 and π_2 as

$$\pi_1 + \pi_2 = \int_0^L \left\{ \begin{array}{l} \frac{1}{2}(G_1J_1 + G_2J_2)[\varphi'(x)]^2 + \frac{1}{2}(E_1I_{w1} + E_2I_{w2})[\varphi''(x)]^2 \\ \frac{1}{2}[E_1I_{y1}[w_1''(x)]^2 + E_2I_{y2}[w_2''(x)]^2] \end{array} \right\} \mathrm{d}x \tag{A.1}$$

In the case of free vibrations, the following system of differential equations is obtained from Eq. (9) corrected by the adding warping terms:

$$\begin{cases} EI_1 w_1^{(4)} - K(w_2 - w_1 - h_0\varphi) + m_1\ddot{w}_1 = 0 \\ EI_2 w_2^{(4)} + K(w_2 - w_1 - h_0\varphi) + m_2\ddot{w}_2 = 0 \\ GJ_0\varphi'' - EI_{w0}\varphi^{(4)} + Kh_0(w_2 - w_1 - h_0\varphi) - m_0 r_0^2\ddot{\varphi} = 0 \end{cases} \quad \text{with } EI_{w0} = E_1I_{w1} + E_2I_{w2} \tag{A.2}$$

Introducing the solution equation (11) in this coupled system of partial-differential equations leads to the calculation of the determinant:

$$\begin{vmatrix} EI_1\left(\frac{n\pi}{L}\right)^4 + K - m_1\Omega^2 & -K & Kh_0 \\ -K & EI_2\left(\frac{n\pi}{L}\right)^4 + K - m_2\Omega^2 & -Kh_0 \\ Kh_0 & -Kh_0 & GJ_0\left(\frac{n\pi}{L}\right)^2 + EI_{w0}\left(\frac{n\pi}{L}\right)^4 + Kh_0^2 - m_0r_0^2\Omega^2 \end{vmatrix} = 0 \tag{A.3}$$

Therefore, previous results can be used by replacing the torsional stiffness GJ_0 by the equivalent term

$$\tilde{G}\tilde{J}_0 = GJ_0 + EI_{w0}\left(\frac{n\pi}{L}\right)^2 \tag{A.4}$$

Finally, the natural frequencies equation is obtained from a generalisation of Eq. (14):

$$\begin{aligned} &\left(EI_1\left(\frac{n\pi}{L}\right)^4 - m_1\Omega^2\right)\left(EI_2\left(\frac{n\pi}{L}\right)^4 - m_2\Omega^2\right)\left(\tilde{G}\tilde{J}_0\left(\frac{n\pi}{L}\right)^2 + Kh_0^2 - m_0r_0^2\Omega^2\right) \\ &\quad + K\left(\tilde{G}\tilde{J}_0\left(\frac{n\pi}{L}\right)^2 - m_0r_0^2\Omega^2\right)\left(EI_0\left(\frac{n\pi}{L}\right)^4 - m_0\Omega^2\right) = 0 \end{aligned} \tag{A.5}$$

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